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RATES OF EXCHANGE.

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The general subject of rates of exchange is, of course, a complicated one. In this paper an elementary discussion of the subject is presented, following Cournot,[1] under the special hypothesis that there is no flow of gold between the various markets involved. Thus the rates of exchange are determined by a system of simultaneous linear homogeneous equations in n unknowns, the ratios of which are the rates of exchange between the n markets. The matrix of this system is of a special type which possesses interesting properties.

1. Preliminary Example. For the sake of clearness consider first a special case in which only three financial markets are involved, *e.g.* New York, London, Paris. Suppose that these form a closed system, *i.e.*, that they have dealings only with each other. We will suppose also that transactions are carried on by exchanges of credit, *i.e.*, that there is no flow of gold from one market to another.

Let the markets in question be indicated by subscripts 1, 2, 3, respectively, and suppose that

New York possesses m_{12} pounds of credit at London,
New York possesses m_{13} francs of credit at Paris,
London possesses m_{21} dollars of credit at New York,
etc.

Suppose also that

One dollar at New York is equal in value to C_{12} pounds at London,
One dollar at New York is equal in value to C_{13} francs at Paris,
etc.

The quantities C_{12}, C_{13}, C_{23}, etc., are called *rates of exchange*. Since there is no flow of gold between markets we must have:

$$\begin{aligned} m_{21} + m_{31} &= \phantom{m_{21}C_{12} + {}} m_{12}C_{21} + m_{13}C_{31}, \\ m_{12} \phantom{{}+ m_{31}} + m_{32} &= m_{21}C_{12} \phantom{{}+ m_{12}C_{21}} + m_{23}C_{32}, \\ m_{13} + m_{23} \phantom{{}+ m_{32}} &= m_{31}C_{13} + m_{32}C_{23}. \end{aligned} \tag{A}$$

The first equation states that the number of dollars of credit which London and Paris together hold at New York is equal to the combined value in dollars at New York of the number of pounds of credit at London and the number of francs of credit at Paris held by New York. The other equations have a similar meaning.

Since a dollar at New York is worth C_{12} pounds of credit at London, and a pound at London is worth C_{23} francs at Paris, it follows that

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[1] Augustin Cournot, *The Mathematical Principles of the Theory of Wealth,* chapter III, New York, 1897. The equations, which are due to Cournot, were discussed in a course given by Professor Evans at the Rice Institute.

$$C_{12}C_{23} = C_{13}.$$

Together with equations (A) the rates of exchange therefore satisfy the equations

$$C_{ij}C_{jk} = C_{ik}, \qquad C_{ii} = 1. \qquad i, j, k = 1, 2, 3. \quad (B)$$

To solve equations (A) and (B) we multiply the first of (A) by C_{11} ($= 1$), the second by C_{21}, the third by C_{31}, thus converting the sums involved into dollars. We get

$$\begin{aligned} (m_{21} + m_{31})C_{11} - m_{12}C_{21} - m_{13}C_{31} &= 0, \\ - m_{21}C_{11} + (m_{12} + m_{32})C_{21} - m_{23}C_{31} &= 0, \\ - m_{31}C_{11} - m_{32}C_{21} + (m_{13} + m_{23})C_{31} &= 0. \end{aligned} \qquad (C)$$

Since the sum of the left-hand members is equal to zero, these equations are linearly dependent. On solving the last two equations we get

$$C_{21} = \frac{\begin{vmatrix} m_{21} & -m_{23} \\ m_{31} & m_{13} + m_{23} \end{vmatrix}}{\begin{vmatrix} m_{12} + m_{32} & -m_{23} \\ -m_{32} & m_{13} + m_{23} \end{vmatrix}} = \frac{\begin{vmatrix} m_{21} + m_{31} & -m_{13} \\ -m_{31} & m_{13} + m_{23} \end{vmatrix}}{\begin{vmatrix} m_{12} + m_{32} & -m_{23} \\ -m_{32} & m_{13} + m_{23} \end{vmatrix}}.$$

Similarly

$$C_{31} = \frac{\begin{vmatrix} m_{21} + m_{31} & -m_{12} \\ -m_{21} & m_{12} + m_{32} \end{vmatrix}}{\begin{vmatrix} m_{12} + m_{32} & -m_{23} \\ -m_{32} & m_{13} + m_{23} \end{vmatrix}}.$$

In general, provided none of the two-rowed determinants involved is equal to zero, the solution of equations (C), satisfying (A) and (B), is

$$C_{ij} = \frac{D_i}{D_j}, \qquad i, j = 1, 2, 3,$$

where D_i is the determinant obtained by striking out the ith row and column of the matrix

$$(\mathbf{m}) = \begin{Vmatrix} m_{21} + m_{31} & -m_{12} & -m_{13} \\ -m_{21} & m_{12} + m_{32} & -m_{23} \\ -m_{31} & -m_{32} & m_{13} + m_{23} \end{Vmatrix}.$$

Since the m's are essentially non-negative, the C_{ij}'s are all positive, for

$$\begin{aligned} D_1 &= m_{12}m_{13} + m_{12}m_{23} + m_{13}m_{32}, \\ D_2 &= m_{23}m_{21} + m_{23}m_{31} + m_{21}m_{13}, \\ D_3 &= m_{31}m_{32} + m_{31}m_{12} + m_{32}m_{21}. \end{aligned}$$

What happens if the rank of ($\mathbf{m}$), which in general is two, is reduced to unity?

If we require that none of the quantities C_{ij} be equal to zero, we find that one of the following three cases must subsist:

$$\begin{array}{ll}
(a) & m_{12} = m_{13} = m_{21} = m_{31} = 0, \\
(b) & m_{23} = m_{21} = m_{32} = m_{12} = 0, \\
(c) & m_{31} = m_{32} = m_{13} = m_{23} = 0.
\end{array}$$

In case (*a*) the three equations (*C*) amount to the single equation:

$$m_{32}C_{21} - m_{23}C_{31} = 0,$$

i.e.,

$$C_{23} = \frac{C_{21}}{C_{31}} = \frac{m_{23}}{m_{32}}.$$

If m_{23} and m_{32} are both different from zero, the rates of exchange between London and Paris are uniquely determined. These two markets have no dealings with New York. Their rates of exchange with New York are indeterminate. The situation here disclosed, which in the case of only three markets is perhaps trivial, is a special example of a more interesting fact in the case of n markets. In fact, it will be proved in general that:

If the rank of the matrix (**m**) *is r, and none of the rates of exchange is zero, then the n markets are divided up into n — r separate groups, each group consisting of a certain number of markets which have dealings with each other but not with any market in any other group. The rate of exchange between any two markets of the same group is uniquely determined. Between two markets of different groups the rate of exchange is indeterminate.*

Let us consider the general case.

2. Let there be n markets, $M_1, M_2, \cdots, M_n$, and suppose that

$$M_i \text{ holds } m_{ij}\ j\text{-units of credit at } M_j,$$

and that

$$\text{one } i\text{-unit at } M_i \text{ is equal in value to } C_{ij}\ j\text{-units at } M_j.$$

The quantity C_{ij} is called the *rate of exchange at* M_i *on* M_j. As before, we suppose that business between the markets is in equilibrium, *i.e.*, there is no flow of gold between them. The following equations are therefore true:

$$\begin{array}{rl}
m_{21}+m_{31}+\cdots+m_{n1} = & m_{12}C_{21}+m_{13}C_{31}+\cdots+m_{1n}C_{n1}, \\
m_{12}+m_{32}+\cdots+m_{n2} = & m_{21}C_{12}+m_{23}C_{32}+\cdots+m_{2n}C_{n2}, \\
m_{13}+m_{23}+\cdots+m_{n3} = & m_{31}C_{13}+m_{32}C_{23}+\cdots+m_{3n}C_{n3}, \\
\cdots\cdots\cdots\cdots & \cdots\cdots\cdots\cdots\cdots\cdots \\
m_{1n}+m_{2n}+\cdots+m_{n-1,n} = & m_{n1}C_{1n}+m_{n2}C_{2n}+\cdots+m_{n,n-1}C_{n-1,n}.
\end{array} \qquad (1)$$

$$C_{ij}C_{jk} = C_{ik}, \qquad i, j, k = 1, 2, \cdots, n. \qquad (2)$$

It should be noted that equations (2) are not independent; also that they contain the equations $C_{ii}^2 = C_{ii}$, which will be interpreted as $C_{ii} = 1$.

If we multiply the first equation of (1) by C_{1p}, the second by C_{2p}, etc., thus converting the amounts of money into currency units of M_p, and if we make use of the relations $C_{ij}C_{jp} = C_{ip}$, we obtain the set of n homogeneous equations:

$$\begin{array}{rrrrrl} \sum_i m_{i1}C_{1p} & - \; m_{12}C_{2p} & - \; m_{13}C_{3p} & - \cdots - & m_{1n}C_{np} & = 0, \\ - \; m_{21}C_{1p} & + \sum_i m_{i2}C_{2p} & - \; m_{23}C_{3p} & - \cdots - & m_{2n}C_{np} & = 0, \\ \cdot \quad \cdot & \cdot \quad \cdot & \cdot \quad \cdot & \cdot \quad \cdot & \cdot \quad \cdot & \cdot \\ - \; m_{n1}C_{1p} & - \; m_{n2}C_{2p} & - \; m_{n3}C_{3p} & - \cdots + & \sum_i m_{in}C_{np} & = 0. \end{array} \tag{3}$$

Here p is one of the numbers $1, 2, \cdots, n$ and $m_{ii} = 0$ is inserted to simplify the notation. Our object is to solve equations (1) in conjunction with (2) and to discuss the effect of the rank of the matrix of (3) upon the solutions.

3. We consider matrices of two kinds, namely:

$$(\mathbf{a}) = \begin{Vmatrix} \sum_i a_{i1} & -a_{12} & -a_{13} & \cdots & -a_{1n} \\ -a_{21} & \sum_i a_{i2} & -a_{23} & \cdots & -a_{2n} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ -a_{n1} & -a_{n2} & -a_{n3} & \cdots & \sum_i a_{in} \end{Vmatrix},$$

$$(\mathbf{b}) = \begin{Vmatrix} \sum_i a_{i1} + b_1 & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & \sum_i a_{i2} + b_2 & \cdots & -a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ -a_{n1} & -a_{n2} & \cdots & \sum_i a_{in} + b_n \end{Vmatrix},$$

in which all of the numbers a_{ij}, b_i are non-negative and $a_{ii} = 0$. The matrix (**a**) is a special case of the matrix (**b**). In general we shall refer to a determinant (matrix) in which each element in the principal diagonal is either greater than or equal to the sum of the absolute values of the other elements in its column, the latter being non-positive numbers, as a *determinant* (*matrix*) of *type* (**b**). The special case in which each principal element is *equal* to the sum of the absolute values of the other elements in its column will be called a *determinant* (*matrix*) of *type* (**a**). The determinant (**a**) is equal to zero.

THEOREM 1. *The value of the determinant* (**b**) *is not negative.*

For the proof by induction, first note that the theorem is evident when (**b**) is of order 1 or 2. If we expand (**b**) as a polynomial in the b_i's, we find that the constant term is precisely the determinant (**a**); the coefficient of b_k is the determinant found by omitting the kth row and column of (**a**)—this is a determinant of type (**b**) of order $n - 1$; similarly the coefficient of $b_k b_l$ is a determinant of type (**b**) of order $n - 2$; in fact the coefficient in every term of the polynomial is a determinant of type (**b**) of order less than n, except the coefficient of $b_1 b_2 \cdots b_n$, which is unity, and the constant term which is the determinant (**a**) and is equal to zero. Thus if the theorem is true when the order is less than n, it is true when the order is equal to n.

THEOREM 2. *If the determinant* (**b**) *is zero, it contains a principal minor of type* (**a**).

If all of the b's are zero, (**b**) itself is of type (**a**). If not all of the b's are zero, the coefficient of any product of non-vanishing b's must be zero, since there are no negative terms in the expansion of (**b**) as a polynomial in the b's. In particular, the product of all the b's must be zero and therefore at least one b is zero. Since the coefficient of any product of b's is a determinant of type (**b**) of lower order than the given determinant, the theorem follows by induction.

If a determinant, of type (**b**) or (**a**), of order n contains no principal minor of type (**a**) of order less than n, it is said to be a *simple* determinant of type (**b**) or (**a**). Two simple minors of type (**a**) are quite separate.

If (**b**) is equal to zero, we can now arrange its rows and columns in a specially convenient order. Let $\alpha_1, \alpha_2, \cdots \alpha_s$ be the simple principal minors of type (**a**) which it contains and let their orders be $k_1, k_2, \cdots k_s$. By shifting rows and columns similarly α_1 can be made to lie in the first k_1 rows and columns, α_2 in the next k_2 rows and columns and so forth. The elements of the last l rows and columns, $l = n - (k_1 + k_2 + \cdots + k_s)$, will form a non-vanishing determinant β of type (**b**). When the rows and columns of (**b**) are arranged in this manner—or when the variables and equations (3) are correspondingly arranged—the arrangement is said to be *compact.*

The following theorems are now easily proved:

THEOREM 3. *If a certain principal minor α of order k, is equal to zero, then every minor of order k taken from the same k columns which contain α is equal to zero.*[1]

THEOREM 4. *Every $(n - 1)$-rowed principal minor of a simple determinant of type* (**a**) *is different from zero.*

COROLLARY. *The rank of a simple determinant of type* (**a**) *is $n - 1$.*

THEOREM 5. *The rank of* (**b**) *is $n - s$, s being the number of simple principal minors of type* (**a**) *which* (**b**) *contains.*

THEOREM 6. *The cofactor of every element of the jth column of the determinant* (**a**) *is equal to D_j, the determinant obtained by striking out the jth row and column.*

4. We can now apply the results of **3** to the solution of equations (1) and (2). The following notation is used:

$$(\mathbf{m}) = \left\|\begin{array}{cccc} \sum_{i=1}^{n} m_{i1} & -m_{12} & \cdots & -m_{1n} \\ -m_{21} & \sum_{i=1}^{n} m_{i2} & \cdots & -m_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ -m_{n1} & -m_{n2} & \cdots & \sum_{i=1}^{n} m_{in} \end{array}\right\| .$$

This matrix is supposed to be compactly arranged. Its determinant is equal

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[1] If a principle minor of order h of (b) is itself of type (a), then all the b's in that minor are zero and all the elements outside of that minor in its h columns are zero, for we have conditions of the form

$$\sum_{i=1}^{n} a_{ij} + b_j = \sum_{i=1}^{h} a_{ij}, \qquad \text{whence} \qquad \sum_{i=h+1}^{n} a_{ij} + b_j = 0,$$

and since there are no negative terms in this polynomial each term must be zero separately.

to zero; $s =$ the number of simple principal minors, $\alpha_1, \alpha_2, \cdots \alpha_s$, of type (**a**). By theorem 5 the rank of (**m**) is $n - s$; $k_1, k_2, \cdots k_s$ are the respective orders of $\alpha_1, \alpha_2, \cdots \alpha_s$; $l =$ the order of the minor determinant β, of type (**b**), lying in the last l rows and columns; $\beta \neq 0$; $k = k_1 + k_2 + \cdots + k_s$; $n = k + l$.

There are three cases to consider:

Case I. (**m**) is of rank $n - 1$, $s = 1$, $k = n$, $l = 0$. (**m**) is simple.

Case II. (**m**) is of rank $n - s$, $s > 1$, $k = n$, $l = 0$. (**m**) contains s simple principal minors of type (**a**) which contain all of the non-zero elements.

Case III. (**m**) is of rank $n - s$, $s \geqq 1$, $k < n$, $l > 0$. There is a principal minor determinant β, lying in the last l rows and columns, of type (**b**) and positive.

Case I. Recalling that $C_{pp} = 1$ $(p = 1, 2, \cdots, n)$, we see that equations (3) have one and only one solution, since the rank of (**m**) is $n - 1$. This solution is:

$$C_{ip} = \frac{D_i}{D_p} > 0,$$

where D_i is the cofactor of every element of the ith column, by theorem 6. This formula gives the only solution of (1) and (2).

Case II. In this case equations (1) consist of s separate sets, each of which is of the kind considered in Case I. In fact, by using equations (2) and multiplying the k_1 equations of the first set in order by the numbers

$$C_{1p_1}, C_{2p_1}, \cdots, C_{k_1p_1},$$

where p_1 is some one of the numbers $1, 2, \cdots, k_1$, we obtain a set of equations of the form (3) of the type discussed in Case I, the unknowns being

$$C_{1p_1}, C_{2p_1}, \cdots, C_{k_1p_1}.$$

Similarly the second set of k_2 equations is reduced to the form (3), the unknowns being

$$C_{k_1+1,p_2}, C_{k_1+2,p_2}, \cdots, C_{k_1+k_2,p_2},$$

where p_2 is one of the numbers $k_1 + 1$, $k_1 + 2$, $\cdots$, $k_1 + k_2$. The remaining sets are treated in the same way. The s sets of equations, having matrices of ranks $k_1 - 1$, $k_2 - 1$, $\cdots$, $k_s - 1$, can be solved uniquely for the quantities C_{ij} when i, j correspond to two markets of the same set. But it is evident that the C_{ij}'s are indeterminate when i, j correspond to markets of different sets.

Case III. In this case it is impossible to obtain a solution which contains no zeros. For, consider the last l equations of (1). If we multiply them in order by the quantities

$$C_{k+1,p}, C_{k+2,p}, \cdots, C_{n,p},$$

we obtain a set of homogeneous equations in these quantities. The determinant

β of the set is different from zero. It follows, therefore, that they are satisfied only by

$$C_{k+1,p} = C_{k+2,p} = \cdots = C_{n,p} = 0;$$

and this result holds when $p = 1, 2, \cdots, k$. If now we substitute these zeros in the first k equations (1), we obtain a set of k equations of the type considered in Case II. We see therefore that it is possible to find values of C_{ij} $[i = 1, 2, \cdots, n;$ $j = 1, 2, \cdots, k]$ which satisfy the first k equations of (1) together with those of equations (2) which do not involve the subscripts $k + 1, k + 2, \cdots, n$. The last l equations (1) can not be satisfied in conjunction with those of equations (2) which involve these subscripts.

From the point of view of economics this result means that there are s separate sets of solvent markets (corresponding to the s simple matrices of type (**a**)) such that no two markets of different sets have any dealings with each other. The rate of exchange between two such markets is indeterminate, but between two markets of the same set it is determined uniquely as in Case II. The l markets not included in these s sets are bankrupt; whereas the credit of each of them at any solvent market is zero, some of them have debit balances with at least one of the solvent markets.

A SIMPLE THEORY OF COMPETITION.

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1. Postulates. Significant theories of economics may be discussed mathematically in very simple terms, and results may be stated, in precise fashion, which a non-mathematical theory could not be expected to deduce at all. Moreover the assumptions are simple enough to be verified approximately in actual societies, and yield results which may be easily calculated in definite cases. Such a rudimentary theory of competition is given below, in terms of linear and quadratic functions, and the elements of the calculus.

Let

$$q(u) = Au^2 + Bu + C \tag{1}$$

represent, as a quadratic function, the total cost of producing (that is, putting on the market) u units of a commodity in unit time. Let

$$y = ap + b \tag{2}$$

represent, as a linear function, the amount of goods, y, which, if the price is p, will be bought in the market in unit time.

Suppose there are two producers, each manufacturing, subject to the same cost function given by (1), amounts u_1 and u_2, respectively, in unit time, and each trying to make his profit a maximum. What will be the amounts u_1 and u_2 manufactured, and the price p at which the goods are sold? This is the problem of competition expressed in simplest terms.